BAULKHAM HILLS HIGH SCHOOL

2016

YEAR 11 December - Task 1

Mathematics Extension 2

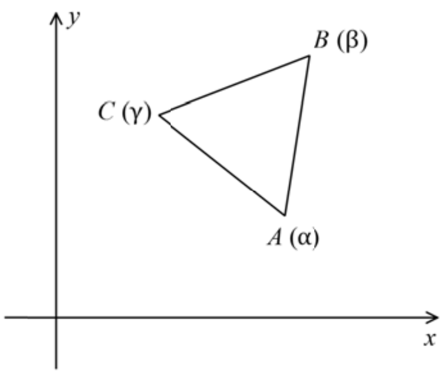
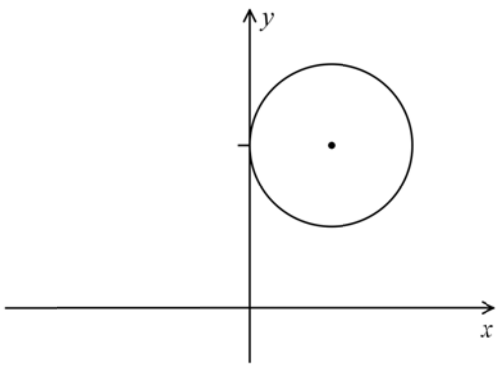
General Instructions

- Reading time – 5 minutes
- Working time – 50 minutes
- Write using non erasable black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions
- Start a new page for each question

Total marks – 35
Exam consists of 3 pages.

Question 1 (11 marks) **Start on the appropriate page of your answer booklet.**		**Marks**
a)	Find :	
	(i) $\lvert 2+3i \rvert$	**1**
	(ii) $\arg(-1-i)$	**1**
	(iii) $\frac{4+3i}{2-i}$	**2**
b)	(i) Find $\sqrt{15-8i}$	**2**
	(ii) Hence solve $z^2-(2+3i)z-5+5i=0$	**3**
c)	Draw a neat sketch of the following locus: $$z\overline{z}-3(z+\bar{z})\leq 0$$	**2**

Question 2 (12 marks) **Start on the appropriate page of your answer booklet.**		**Marks**
a)	(i) If $w=\frac{1-i\sqrt{3}}{2}$, show $w^3=-1$.	**2**
	(ii) Hence calculate w^{11}.	**2**
b)	If $z=\cos\theta+i\sin\theta$	
	i) Given that $(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$ show that $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$.	**3**
	ii) Hence show that polynomial equation $16x^5-20x^3+5x-1=0$ has roots $\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$ and 1.	**2**
	iii) Hence deduce that $\cos\frac{2\pi}{5}\times\cos\frac{4\pi}{5}=-\frac{1}{4}$.	**3**

Question 3 (12 marks) Start on the appropriate page of your answer booklet.		Marks
a)	Find the equation of the locus of z, stating any restrictions, such that: $$\arg\left[\frac{z-1+2i}{z-3+2i}\right]=\frac{\pi}{4}$$	3
b)	Let $w=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$	
	(i) Show that $w^3=1$.	1
	(ii) Show that $1+w+w^2=0$.	1
	The points A, B and C form an equilateral triangle in the Argand diagram, and represent the complex numbers α, β, γ respectively. 	
	(iii) Show that $\alpha-\gamma=w(\gamma-\beta)$.	2
	(iv) Deduce that $\alpha+w\beta+w^2\gamma=0$.	1
c)	The locus of z, such that $\lvert z-1-2i\rvert=1$ is sketched below. 	
	(i) Find the maximum value of $\lvert z+1\rvert$.	2
	(ii) Find the maximum value of $\arg(z+1)$ correct to the nearest degree.	2

End of Exam

X2 - solutions - $\boxed{\text{Dec. 2016}}$

Question 1 (11 marks)

	Marks
a) Find	
(i) $\|2+3i\| = \sqrt{2^2+3^2} = \sqrt{13}$	1 - correct solns
(ii) $\text{Arg}(-1-i) = -\frac{3\pi}{4}$	1 - correst solns
iii) $\frac{4+3i}{2-i} \times \frac{2+i}{2+i} = \frac{8-3+i(6+4)}{4+1}$	2 - correct soln.
$= \frac{5+10i}{5} = 1+2i$	1 - multiplies by conjugate
b) i) Let $\sqrt{15-8i} = z$ where $z = x+iy$	
$\therefore 15-8i = z^2 = x^2-y^2+2xyi$	2 - correct solns
Compare real & imaginary parts	
$\therefore$ (1) $15 = x^2-y^2$	1 - attempts to solve simult. equations by equating real & imaginary parts
(2) $-8 = 2xy \therefore x = -\frac{4}{y}$	
$\therefore 15 = \left(\frac{-4}{y}\right)^2 - y^2$	
$\therefore y^4 + 15y^2 - 16 = 0$	
$(y^2+16)(y^2-1) = 0$	
$y^2+16 = 0$ no soln. for y-real; $y = \pm 1$	

$$\left.\begin{array}{l} \therefore y = 1 \quad \therefore x = -4 \\ y = -1 \quad \therefore x = 4 \end{array}\right\} \therefore \sqrt{15-8i} = \begin{cases} -4+i \\ 4-i \end{cases}$$

or $\sqrt{15-8i} = \pm(4-i)$

Question 1 b) cont. (3 marks)

Marks

ii) $z^2 - (2+3i)z - 5 + 5i = 0$

$$z = \frac{2+3i \pm \sqrt{[-(2+3i)]^2 - 4\times(-5+5i)}}{2}$$

3 - correct solns.

$$z = \frac{2+3i \pm \sqrt{4+12i-9+20-20i}}{2}$$

2 - manipulates quad. formula to achieve $\Delta = 15-8i$

$$z = \frac{2+3i \pm \sqrt{15-8i}}{2}$$

from part i) $z = \dfrac{2+3i \pm (\pm(4-i))}{2}$

1 - substitutes correctly into quadratic formula

$$\therefore z = \frac{2+3i \pm (4-i)}{2}$$

$$z = \frac{6+2i}{2} = 3+i$$

$$z = \frac{-2+4i}{2} = -1+2i$$

Question 1c) (2 marks)

$z\bar{z} - 3(z+\bar{z}) \leq 0$

Marks

let $z = x + iy$ $\quad z\bar{z} = x^2 + y^2$

$z + \bar{z} = 2\,\text{Re}(z) = 2x$

2 - correct sketch

$\therefore x^2 + y^2 - 3(2x) \leq 0$

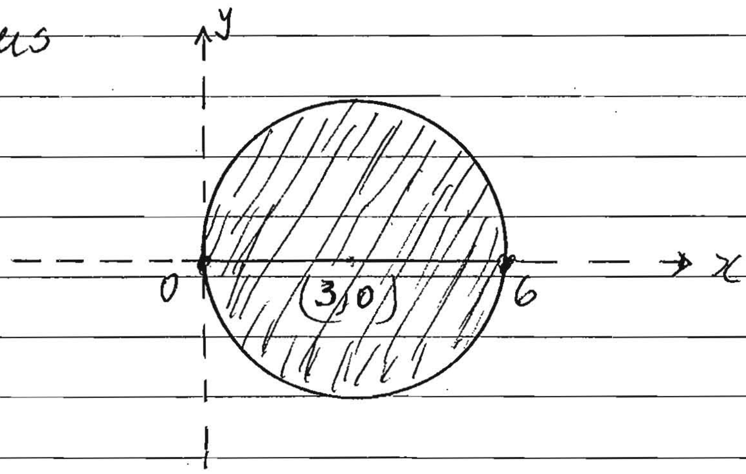
1 - finding equation of the circle

$\therefore x^2 - 6x + 9 + y^2 \leq 0 + 9$

$\therefore (x-3)^2 + y^2 \leq 9$

$\therefore$ boundary is a circle centre $(3,0)$ radius 3

locus



Question 2 (12 marks)

a) i) if $w = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

$$w = |w| \operatorname{cis}\theta$$

$$|w| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(-\tfrac{\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1} \frac{\sqrt{3}/2}{1/2} = -\frac{\pi}{3}$$

$$\therefore w = 1 \times \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

$$\therefore w^3 = 1^3 \operatorname{cis}\left[3 \times -\frac{\pi}{3}\right]$$

$$= \operatorname{cis}(-\pi) = -1 \text{ as shown.}$$

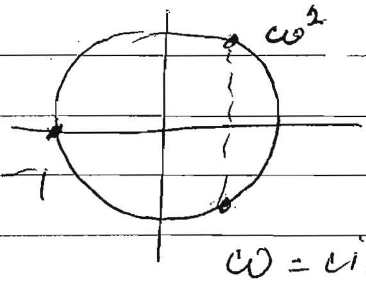
ii) $\omega^{11} = (w^3)^3 \times \omega^2$

from (i) $\omega^{11} = (-1)^3 \times \omega^2 = -\omega^2 = -\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

$\omega^{11} = +\operatorname{cis}\frac{\pi}{3}$ or

$\omega^{11} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

OR since ω is one of the complex roots of $\omega^3 = -1$



$\omega = \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$\therefore$ the other roots are $\omega = -1$

and $\omega^2 = \overline{\omega} = \operatorname{cis}\frac{\pi}{3}$

$\therefore \omega^{11} = \omega^2 = \operatorname{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Marks

2- correct working

1-expressing w in mod/arg. form

2- correct solns.

1- manipulates $\omega^{11} = (\omega^3)^3 \times \omega^2$ correctly

Question 2 - cont.

2b) i) If $z = \cos\theta + i\sin\theta$

show $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

since $z = \cos\theta + i\sin\theta$

$\therefore z^5 = (\cos\theta + i\sin\theta)^5$ & $z^5 = \cos 5\theta + i\sin 5\theta$

Now: $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta$
$+ 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$

and since $z^5 = \cos 5\theta + i\sin 5\theta$ (D'Maivres theorem)

Now equate real parts of both expansions

$\therefore \cos 5\theta = \cos^5\theta + 10\cos^3\theta\, i^2\sin^2\theta + 5\cos\theta\, i^4\sin^4\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ $\therefore$ shown

Marks

3 - correct working

2 - correctly uses De Moivre's theorem and "equates real parts"

1 - expands binomial exp. and simplifies correctly

Questio 2 b) cont.

Marks

(ii) let $x = \cos\theta$

~~since~~ $16x^5 - 20x^3 + 5x - 1 = 0$

$\therefore 16x^5 - 20x^3 + 5x = 1$

which is $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 1$

~~from~~ i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$\therefore$ equation to solve $\cos 5\theta = 1$

So the roots of $16x^5 - 20x^3 + 5x - 1 = 0$

are the same as roots of $\cos 5\theta = 1$

which are $\therefore 5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

but $x = \cos\theta \therefore x = 1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}$ and $\cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$

$\therefore$ shown

2 - correct explanation

1 - solves $\cos 5\theta = 1$ for θ.

(iii) Deduce $\cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} = -\frac{1}{4}$

soln: From ii) Roots of $16x^5 - 20x^3 + 5x - 1 = 0$

are $\alpha = 1, \beta = \cos\frac{2\pi}{5}, \gamma = \cos\frac{4\pi}{5}, \delta = \cos\frac{6\pi}{5}, \varphi = \cos\frac{8\pi}{5}$

Now product of roots

$$\alpha\beta\gamma\delta\varphi = -\frac{e}{a} = -\frac{-1}{16}$$

$$\therefore 1 \times \cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} \times \cos\frac{6\pi}{5} \times \cos\frac{8\pi}{5} = \frac{1}{16}$$

but $\cos\frac{2\pi}{5} = \cos\left(2\pi - \frac{2\pi}{5}\right) = \cos\frac{8\pi}{5}$

and $\cos\frac{4\pi}{5} = \cos\left(2\pi - \frac{4\pi}{5}\right) = \cos\frac{6\pi}{5}$

$$\therefore 1 \times \cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} \times \cos\frac{4\pi}{5} \times \cos\frac{2\pi}{5} = \frac{1}{16}$$

PTO

3 - correct explanation

2 - expresses the product of roots in terms of $\cos\frac{2\pi}{5}$ & $\cos\frac{4\pi}{5}$ with correct justification

1 - calculates the product of roots of eqn. from ii)

Q.2 b ii) cont.

$$\therefore \cos^2\frac{2\pi}{5} \times \cos^2\frac{4\pi}{5} = \frac{1}{16}$$

$$\therefore \cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} = \pm\sqrt{\frac{1}{16}} = \pm\frac{1}{4}$$

but since $\cos\frac{2\pi}{5} > 0$ and $\cos\frac{4\pi}{5} < 0$

$$\therefore \cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} < 0$$

$$\therefore \cos\frac{2\pi}{5} \times \cos\frac{4\pi}{5} = -\frac{1}{4}$$

Question 3 (12 marks)

Marks

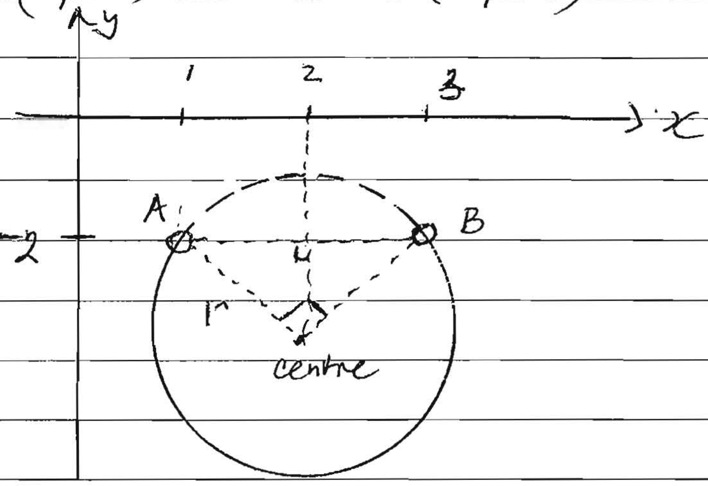
a) $\arg\left[\frac{z-1+2i}{z-3+2i}\right] = \arg\left[\frac{z-(1-2i)}{z-(3-2i)}\right] = \frac{\pi}{4}$

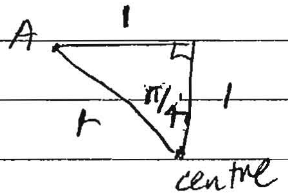
3 - correct solns.

let $A(1,-2)$ and $B(3,-2)$

2 - finds the correct equation of circle

1 - correct sketch with open circles at $(1,-2)$ & $(3,-2)$





$\therefore r = \sqrt{2}$

centre $(2, -2-1) = (2, -3)$

equation of the circle (full)

$$(x-2)^2 + (y+3)^2 = 2$$

$\therefore$ Locus of z is the major arc of this circle exluding $A(1,-2)$ & $B(3,-2)$

Now $\boxed{\text{with restrictions}}$:

it's a part of circle $(x-2)^2 + (y+3)^2 = 2$ where $y < -2$

or circle $(x-2)^2 + (y+3)^2 = 2$ where $-3-\sqrt{2} \le y < -2$

Question 3 b)

Let $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

Marks

i) $w^3 = \cos\left(3\times\frac{2\pi}{3}\right) + i\sin\left(3\times\frac{2\pi}{3}\right)$

$= \cos 2\pi + i\sin 2\pi = +1$

1 - correct explanation

ii) Now since $\omega^3 = 1$

$\therefore \omega^3 - 1 = 0$

$\therefore (\omega - 1)(\omega^2 + \omega + 1) = 0$

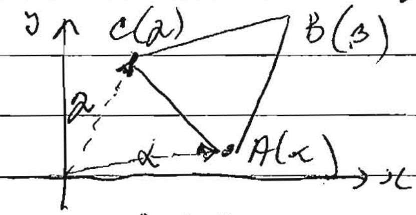
but $\omega \neq 1$ since it's a complex root $\therefore \omega^2 + \omega + 1 = 0$

1 - correct solns with reasoning

OR by sum of roots of $\omega^3 - 1 = 0$

OR manipulates $1 + \omega + \omega^2 = 1 + cis\frac{2\pi}{3} + cis\frac{4\pi}{3} = \ldots$

(iii) Show that $\alpha - \gamma = \omega(\gamma - \beta)$.



since ΔABC is equilateral $\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{AC}|$

Now $\alpha - \gamma = \overrightarrow{CA}$ and $\gamma - \beta = \overrightarrow{BC}$

since $\omega = cis\frac{2\pi}{3}$ and $|\omega| = 1$

$\therefore \omega(\gamma - \beta) = \omega \times \overrightarrow{BC} = \overrightarrow{CA}$

which is rotating $\overrightarrow{BC}$ by $\frac{2\pi}{3}$ anticlockwise

2 - correct working

1 - progress towards solns. by rotating vectors

- manipulates $\alpha - \gamma = \overrightarrow{CA}$ or $\gamma - \beta = \overrightarrow{BC}$ by multiplying by ω

OR $\alpha - \gamma = \overrightarrow{CA} = \overrightarrow{CB} \times cis\left(\frac{-\pi}{3}\right)$

$= \overrightarrow{BC} \times cis\pi \times cis\left(\frac{-\pi}{3}\right)$

$= \overrightarrow{BC} \times cis\left(\frac{2\pi}{3}\right) = \overrightarrow{BC} \times \omega = (\gamma - \beta) \times \omega \therefore$ proven

3b) iv)

Deduce $\alpha + \omega\beta + \omega^2\gamma = 0$

Marks

Solns: From iii) $\alpha - \gamma = \omega(\gamma - \beta)$

$\therefore \alpha - \gamma = \omega\gamma - \omega\beta$

$\therefore \alpha - \gamma - \omega\gamma + \omega\beta = 0$

$\therefore \alpha + \omega\beta - \gamma(1 + \omega) = 0$

but from (ii) $1 + \omega + \omega^2 = 0 \therefore 1 + \omega = -\omega^2$

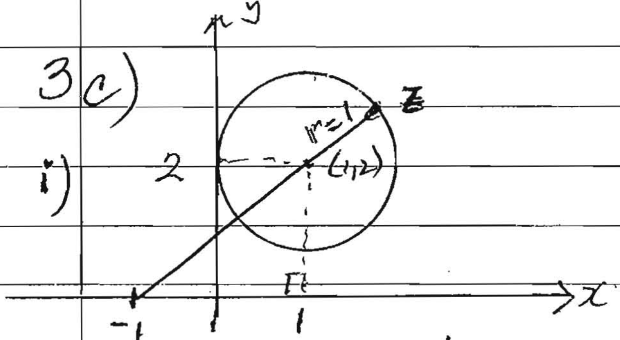
$\therefore \alpha + \omega\beta - \gamma(-\omega^2) = 0$

$\therefore \alpha + \omega\beta + \omega^2\gamma = 0 \therefore$ shown

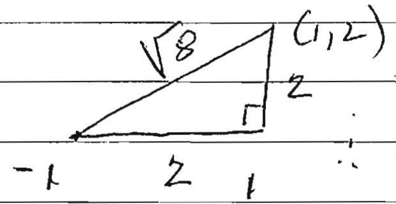
1 - correct solution

3c)

i)



The locus of $|z - (1+2i)| = 1$ is a circle centre $(1,2)$, radius $= 1$



$\therefore$ Max. of $|z+1| = \sqrt{8} + 1$

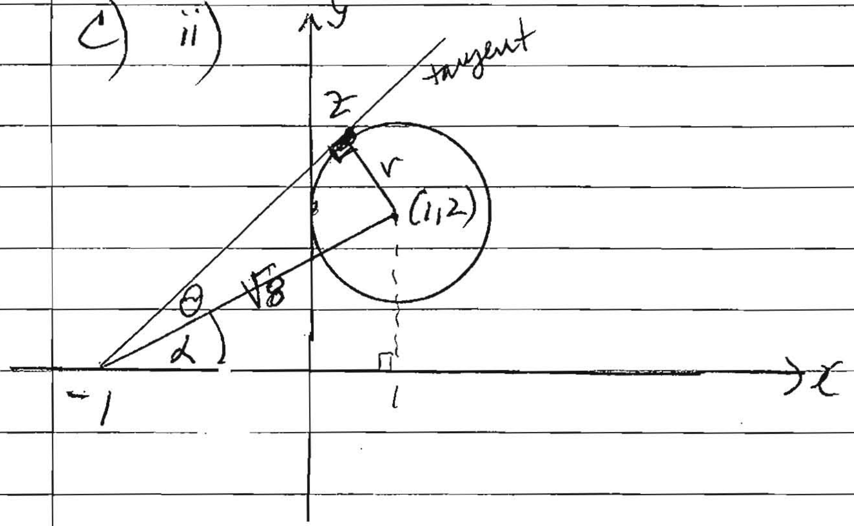
2 - correct solution

1 - finds distance from $(-1,0)$ to centre of circle $(1,2)$

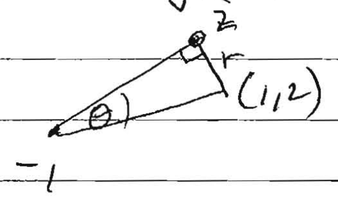
- recognises z with Max. value $|z+1|$ to be intersection of line joining $(-1,0)$ & $(1,2)$ & circle

Question 3

c) ii)



$\alpha = \arg(1+2i-(-1)) = \tan^{-1}\frac{2}{2} = \frac{\pi}{4} = 45°$



$\sin\theta = \frac{r}{\sqrt{8}} = \frac{1}{\sqrt{8}}$

$\therefore \theta = \sin^{-1}\frac{1}{\sqrt{8}} = 20.7°$

$\therefore$ max. value of $\arg(z+1) = 45° + 20.7°$

$\doteqdot 66°$ (nearest degree)

Marks

2 - correct solution

1 - finds arg. of vector of (1,2) from (-1,0)

$\alpha = \arg(1+2i-(-1))$

1 - finds angle θ subtended by tangent from (-1,0) to circle and line joining the centre & (-1,0).